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Evolution of large-scale perturbations in quintessence models

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We carry out a comprehensive study of the dynamics of large-scale perturbations in quintessence scenarios. We model the contents of the Universe by a perfect fluid with equation of state w_f and a scalar field Q with potential $V(Q)$. We are able to reduce the perturbation equations to a system of four first-order equations. During each of the five main regimes of quintessence field behavior, these equations have constant coefficients, enabling an analytic solution of the perturbation evolution by eigenvector decomposition. We determine these solutions and discuss their main properties.

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I. INTRODUCTION

Recent observations seem to indicate that the Universe is undergoing a period of accelerated expansion [1]. Whereas cosmologists initially introduced a cosmological constant in order to explain this, a range of different models have emerged, amongst which quintessence has been particularly prominent in the literature [2,3]. It is defined as a scalar field rolling down its potential and presently dominating the dynamics of the Universe. An important class of quintessence models are known as tracking models [2,3], where the late-time evolution of the field has an attractor behavior rendering its evolution fairly independent of initial conditions. In contrast with a cosmological constant, which is by definition perfectly homogeneous, the quintessence field can, and indeed must, have perturbations.

The evolution of perturbations in quintessence models has been studied by many authors [2,4–6]. In this paper we carry out an exhaustive and elegant analysis of those in the large-scale approximation. We model the contents of the Universe by a perfect fluid with equation of state w_f and a scalar field Q with potential $V(Q)$. We assume a flat Universe throughout.

II. BACKGROUND EVOLUTION

Before studying the perturbations, we recall some results for the homogeneous background evolution. The geometry of the Universe is described by a flat Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2. \quad (1)$$

The Einstein equations

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3m_{\text{Pl}}^2}(\rho_f + \rho_Q), \quad (2)$$

$$2\dot{H} + 3H^2 = 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\frac{8\pi}{m_{\text{Pl}}^2}(p_f + p_Q) \quad (3)$$

relate the matter components to the geometry. The indices “f” and “Q” always refer to the perfect fluid and the quin-

tessence field respectively, and dots are time derivatives. We will use a prime to denote a derivative with respect to $N \equiv \log(a/a_0)$.

The evolution of the fluid is straightforward, with its energy density scaling as $a^{-3(1+w_f)}$, where w_f is the ratio of pressure to energy density of the fluid. The quintessence field follows the Euler-Lagrange equation

$$\ddot{Q} = -3H\dot{Q} - \frac{dV}{dQ} \quad (4)$$

and its equation of state is

$$w_Q \equiv \frac{p_Q}{\rho_Q} = \frac{\dot{Q}^2/2 - V(Q)}{\dot{Q}^2/2 + V(Q)}. \quad (5)$$

Depending on the precise model and on the choice of initial conditions, the quintessence dynamics can feature up to five main regimes, which were classified by Brax *et al.* [5] and which appear in a sequential order. During the first three the quintessence field is subdominant.¹ The “kinetic” regime is characterized by the domination of the kinetic energy which scales as a^{-6} . In the “transition” and “potential” regimes the potential energy dominates and the energy density remains constant. The sound speed of the quintessence field is defined by

$$c_{sQ}^2 \equiv \frac{\dot{p}_Q}{\dot{\rho}_Q} = w_Q - \frac{w'_Q}{3(1+w_Q)} = 1 + \frac{2}{3} \frac{dV/dQ}{H\dot{Q}}, \quad (6)$$

and is equal to 1 or $-2 - w_f$ respectively during those regimes. During the “tracker” regime the quintessence field approximately mimics the behavior of the fluid, and usually its energy density is still subdominant when tracking begins. If $w_Q = w_f$ there is perfect tracking, even if the quintessence field is not subdominant. Finally the field enters its “domination” phase, during which w_Q tends to -1 in most cases.

¹Actually, one can consider initial conditions with domination of the quintessence field, but they usually lead to an “overshoot” of the required present energy density, and are therefore not interesting.

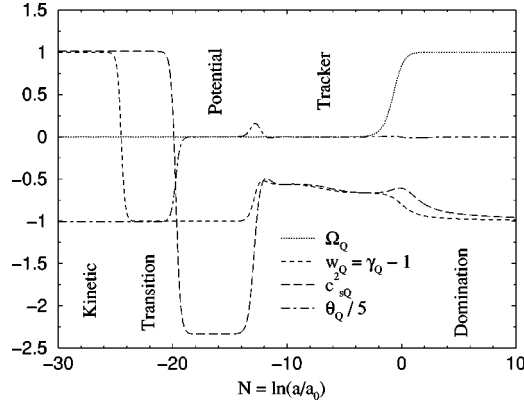


FIG. 1. An example of the evolution of the four parameters Ω_Q , w_Q , c_{sQ}^2 , and $\theta_Q/5$ in a realistic Universe, showing the five different regimes and also the transition between radiation domination and matter domination (at $N \approx -7$). We used an inverse power-law potential $V(Q) = V_0(Q/m_{\text{Pl}})^{-\alpha}$, with $\alpha=1$ and $V_0 = 3 \times 10^{-124} m_{\text{Pl}}^4$. In this case we have the usual sub-dominant tracker regime.

Four useful parameters can be defined to describe these five regimes. They are the quintessence density parameter Ω_Q , the equation of state conveniently parametrized as $\gamma_Q \equiv 1 + w_Q$, the speed of sound c_{sQ}^2 as defined in Eq. (6), and one further parameter relating to the speed of sound defined as

$$\theta_Q \equiv \frac{(c_{sQ}^2)'}{1 - c_{sQ}^2} = -3(1 + c_{sQ}^2) - \frac{d}{dN} \log(Q' dV/dQ). \quad (7)$$

In order to simplify the notation, we define the vector $\mathbf{x} \equiv (\Omega_Q, \gamma_Q, c_{sQ}^2, \theta_Q)$. In Fig. 1, we show an example of the evolution of those parameters. We take the realistic case $w_f = 1/3$ during radiation domination and $w_f = 0$ during matter domination, and we use an inverse power-law quintessence potential. We clearly see the five different regimes, and also the transition between radiation domination and matter domination which in this case occurs during the tracking regime. In Table I we give the values of the parameters in the general case for each regime.

III. PERTURBATION EVOLUTION

In order to describe the perturbations we choose the conformal Newtonian gauge [7]. As long as there is no anisotropic stress, the perturbed metric is

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(t)(1 - 2\Phi)d\mathbf{x}^2, \quad (8)$$

TABLE I. Values of the four parameters Ω_Q , γ_Q , c_{sQ}^2 , and θ_Q during the different regimes.

	Ω_Q	γ_Q	c_{sQ}^2	θ_Q
Kinetic	0	2	1	$-3(3 + w_f)/2$
Transition	0	0	1	$-3(3 + w_f)/2$
Potential	0	0	$-2 - w_f$	0
Usual Tracker	0	γ_Q	w_Q	0
Perfect Tracker	Ω_Q	γ_f	w_Q	0
Domination	1	γ_Q	w_Q	0

where in this case Φ is equal to the gauge-invariant potential defined as in Ref. [7]. We work in Fourier space and compute the first-order perturbed Einstein equations

$$-3H(H\Phi + \dot{\Phi}) - \frac{k^2}{a^2}\Phi = 4\pi G(\delta\rho_f + \delta\rho_Q), \quad (9)$$

$$\ddot{\Phi} + 4H\dot{\Phi} + (2\dot{H} + 3H^2)\Phi = 4\pi G(\delta p_f + \delta p_Q), \quad (10)$$

where the perturbed fluid pressure $\delta p_f = w_f \delta\rho_f$ and the perturbations in the quintessence energy density and pressure are given by

$$\delta\rho_Q = \dot{Q}\delta\dot{Q} - \dot{Q}^2\Phi + \frac{dV}{dQ}\delta Q, \quad (11)$$

$$\delta p_Q = \dot{Q}\delta\dot{Q} - \dot{Q}^2\Phi - \frac{dV}{dQ}\delta Q. \quad (12)$$

The perturbed Euler-Lagrange equation and fluid conservation equation lead to

$$\delta\ddot{Q} + 3H\delta\dot{Q} + \frac{k^2}{a^2}\delta Q + \frac{d^2V}{dQ^2}\delta Q = 4\dot{Q}\dot{\Phi} - 2\frac{dV}{dQ}\Phi, \quad (13)$$

$$\dot{\delta}_f - 3(1 + w_f)\dot{\Phi} = -(1 + w_f)\frac{k}{a}\mathcal{V}_f, \quad (14)$$

where $\delta_f \equiv \delta\rho_f/\rho_f$ and \mathcal{V}_f gives the fluid velocity. From now on we use N as a time variable and study the evolution of Φ , δ_f , $\delta_Q \equiv \delta\rho_Q/\rho_Q$ and $\delta_P \equiv \delta p_Q/\rho_Q$ in the long-wavelength limit ($k/aH \ll 1$). We define the vector $\mathbf{y} \equiv (\Phi, \delta_f, \delta_Q, \delta_P)^T$, and considerable algebra leads to the expression

$$\mathbf{y}' = \mathcal{F}(\gamma_f, \mathbf{x}) \times \mathbf{y}, \quad (15)$$

where the matrix $\mathcal{F}(\gamma_f, \mathbf{x})$ is given by

$$\mathcal{F}(\gamma_f, \mathbf{x}) = \begin{pmatrix} -1 & -\Omega_f/2 & -\Omega_Q/2 & 0 \\ -3\gamma_f & -3\gamma_f\Omega_f/2 & -3\gamma_f\Omega_Q/2 & 0 \\ -3\gamma_Q & -3\gamma_Q\Omega_f/2 & -3\gamma_Q\Omega_Q/2 + 3w_Q & -3 \\ -3\gamma_Qc_{sQ}^2 & -3\gamma_Q\Omega_f/2 & 3\gamma_f\Omega_f/2 + \theta_Q & -3\gamma_Q\Omega_Q/2 + 3w_Q - 3\gamma_f\Omega_f/2 - \theta_Q - 3c_{sQ}^2 \end{pmatrix}. \quad (16)$$

In order to display the above expression in compact form we used the variables $\Omega_f = 1 - \Omega_Q$ and $w_Q = \gamma_Q - 1$, but the matrix depends only on the five independent parameters γ_f , Ω_Q , γ_Q , c_{sQ}^2 and θ_Q . Recall that γ_f and γ_Q take on values between 0 and 2.

The general evolution is of course very complicated, but as seen in Fig. 1, during each of the five regimes described above the coefficients within the matrix take on constant values, and this allows us to study the main features of the perturbation evolution. We are interested in the eigenvalues n_i of the matrix \mathcal{F} and their corresponding eigenvectors \mathbf{y}_i . The solution then takes the form

$$\mathbf{y} = \sum_{i=1}^4 A_i \mathbf{y}_i \exp(n_i N), \quad (17)$$

where the A_i are constants given by the initial conditions.

A. The adiabatic case

Before considering the general case, we restrict ourselves to adiabatic perturbations. For perturbations to be adiabatic, they must share a common perturbation according to the prescription

$$\frac{\delta \rho_f}{\dot{\rho}_f} = \frac{\delta \rho_Q}{\dot{\rho}_Q} = \frac{\delta p_Q}{\dot{p}_Q}, \quad (18)$$

which ensures that all matter perturbations vanish on uniform-density hypersurfaces. Note that the quintessence pressure perturbation, as well as its density perturbation, must satisfy the adiabatic condition (for the perfect fluid adiabaticity of its pressure perturbation is automatically guaranteed by its equation of state). These conditions can be rewritten as

$$\frac{\delta_f}{\gamma_f} = \frac{\delta_Q}{\gamma_Q} = \frac{\delta p}{c_{sQ}^2 \gamma_Q}. \quad (19)$$

It is well known that initially adiabatic perturbations remain purely adiabatic, and indeed it is not difficult to check that these conditions are conserved through evolution by our equations. We can therefore reduce the dynamical system in the adiabatic case to a system of two first-order equations. We define the vector $\mathbf{z} \equiv (\Phi, \delta_f)^T$, and using Eqs. (15) and (19) we find

$$\mathbf{z}' = \mathcal{G}(\gamma_f, \Omega_Q, \gamma_Q) \times \mathbf{z}, \quad (20)$$

where the matrix $\mathcal{G}(\gamma_f, \Omega_Q, \gamma_Q)$ is given by

$$\mathcal{G}(\gamma_f, \Omega_Q, \gamma_Q) = \begin{pmatrix} -1 & -\gamma_{\text{tot}}/2 \gamma_f \\ -3 \gamma_f & -3 \gamma_{\text{tot}}/2 \end{pmatrix}, \quad (21)$$

where $\gamma_{\text{tot}} = \gamma_f \Omega_f + \gamma_Q \Omega_Q$. The eigenvalues and eigenvectors are given in the upper part of Table II. We see, as is well

TABLE II. Top: eigenvalues and eigenvectors of the matrix \mathcal{G} (adiabatic case). Bottom: eigenvalues and eigenvectors of the matrix \mathcal{F} according to the different regimes; $n_{\text{ut}1}$, $n_{\text{ut}2}$, $n_{\text{ut}3}$, $n_{\text{pt}1}$ and $n_{\text{pt}2}$ are given in Eqs. (22)–(26).

Adiabatic perturbations	
$n_1 = 0$	$\mathbf{z}_1 = (-\gamma_{\text{tot}}/2, \gamma_f)$
$n_2 = -1 - 3 \gamma_{\text{tot}}/2$	$\mathbf{z}_2 = (1/3, \gamma_f)$
Kinetic regime	
$n_1 = 0$	$\mathbf{y}_1 = (-\gamma_f/2, \gamma_f, 2, 2)$
$n_2 = -1 - 3 \gamma_f/2$	$\mathbf{y}_2 = (1/3, \gamma_f, 2, 2)$
$n_3 = +6$	$\mathbf{y}_3 = (0, 0, 1, -1)$
$n_4 = 0$	$\mathbf{y}_4 = (0, 0, 1, 1)$
Transition regime	
$n_1 = 0$	$\mathbf{y}_1 = (-1/2, 1, 0, 0)$
$n_2 = -1 - 3 \gamma_f/2$	$\mathbf{y}_2 = (1/3, \gamma_f, 0, 0)$
$n_3 = 0$	$\mathbf{y}_3 = (0, 0, 1, -1)$
$n_4 = -6$	$\mathbf{y}_4 = (0, 0, 1, 1)$
Potential regime	
$n_1 = 0$	$\mathbf{y}_1 = (-1/2, 1, 0, 0)$
$n_2 = -1 - 3 \gamma_f/2$	$\mathbf{y}_2 = (1/3, \gamma_f, 0, 0)$
$n_3 = 0$	$\mathbf{y}_3 = (0, 0, 1, -1)$
$n_4 = -3 + 3 \gamma_f/2$	$\mathbf{y}_4 = (0, 0, -2, \gamma_f)$
Usual tracker regime	
$n_1 = 0$	$\mathbf{y}_1 = (-\gamma_f/2, \gamma_f, \gamma_Q, w_Q \gamma_Q)$
$n_2 = -1 - 3 \gamma_f/2$	$\mathbf{y}_2 = (1/3, \gamma_f, \gamma_Q, w_Q \gamma_Q)$
$n_3 = n_{\text{ut}1} + n_{\text{ut}3}$	$\mathbf{y}_3 = (0, 0, n_{\text{ut}2} + n_{\text{ut}3}, 3 \gamma_f/2)$
$n_4 = n_{\text{ut}1} - n_{\text{ut}3}$	$\mathbf{y}_4 = (0, 0, n_{\text{ut}2} - n_{\text{ut}3}, 3 \gamma_f/2)$
Perfect tracker regime	
$n_1 = 0$	$\mathbf{y}_1 = (-\gamma_f/2, \gamma_f, \gamma_f, w_f \gamma_f)$
$n_2 = -1 - 3 \gamma_f/2$	$\mathbf{y}_2 = (1/3, \gamma_f, \gamma_f, w_f \gamma_f)$
$n_3 = n_{\text{pt}1} + n_{\text{pt}2}$	$\mathbf{y}_3 = \dots$
$n_4 = n_{\text{pt}1} - n_{\text{pt}2}$	$\mathbf{y}_4 = \dots$
Domination regime	
$n_1 = 0$	$\mathbf{y}_1 = (-\gamma_Q/2, \gamma_f, \gamma_Q, w_Q \gamma_Q)$
$n_2 = -1 - 3 \gamma_Q/2$	$\mathbf{y}_2 = (1/3, \gamma_f, \gamma_Q, w_Q \gamma_Q)$
$n_3 = 0$	$\mathbf{y}_3 = (0, 1, 0, 0)$
$n_4 = -3 + 3 \gamma_Q/2$	$\mathbf{y}_4 = (1/3, \gamma_f, 1/3 - w_Q, -\gamma_Q/3)$

known, that there are always a constant and a decaying adiabatic mode, the former giving the late-time solution $\Phi = -\gamma_{\text{tot}} \delta_f/2 \gamma_f = -\delta_{\text{tot}}/2$, where $\delta_{\text{tot}} \equiv \delta \rho_{\text{tot}}/\rho_{\text{tot}}$.

B. The general case

We now return to the full set of perturbation equations (15), continuing to consider the regimes in each of which the coefficients of the matrix \mathcal{F} remain constant. We summarize our main results in the lower part of Table II. For the perfect tracker regime the eigenvectors \mathbf{y}_3 and \mathbf{y}_4 have not been given, since they are long and complicated formulas which are anyway not very relevant. In order to simplify some expressions we have used the variables

$$n_{\text{ut}1} = -\frac{3}{4}(\gamma_f - 2w_Q), \quad (22)$$

$$n_{\text{ut}2} = \frac{3}{4}(\gamma_f + 2w_Q), \quad (23)$$

$$n_{\text{ut}3} = \frac{3}{4}\sqrt{(2w_Q + \gamma_f)^2 - 8\gamma_f}, \quad (24)$$

$$n_{\text{pt}1} = -\frac{3}{4}(2 - \gamma_f), \quad (25)$$

$$n_{\text{pt}2} = \frac{3}{4}\sqrt{(2 - \gamma_f)(2 - \gamma_f - 8\gamma_f\Omega_f)}. \quad (26)$$

For each regime the adiabatic modes can easily be identified as the first two entries in the table. Now let us analyze the two other, nonadiabatic, modes. In the kinetic case there are a growing mode, for which $\delta_P = -\delta_Q$ and thus $\delta\dot{Q} = 0$ (since $\Phi = 0$), and another constant mode corresponding to $\delta Q = 0$. During the transition and potential regimes the former growing mode becomes constant and for each regime the fourth mode is decaying. Therefore, before entering the tracker the quintessence field may feature large nonadiabatic perturbations. As long as the Universe is dominated by the fluid, they are isocurvature perturbations.

In the usual tracker case the last two eigenvalues may have an imaginary part, leading to oscillations, and their real part can either be negative or positive according to the value of γ_f and γ_Q . The properties of the eigenvalues are shown in Fig. 2. However, since the quintessence field has to dominate at the present epoch, ρ_Q must decrease more slowly than ρ_f , and hence $\gamma_Q < \gamma_f$. As one can see in Fig. 2, this implies $\text{Re}(n_{3,4}) < 0$. In the perfect tracker case one easily sees that the last two modes decay, possibly oscillating as they do. As long as all the other modes are decaying, during the tracker regime the constant adiabatic mode y_1 is an attractor. As a result, a long tracker period implies the suppression of all nonadiabatic modes [5,6]. Moreover, in the case of a subdominant tracker the late-time evolution of the perturbations is even independent of the quintessence field initial conditions. Finally, during the domination regime, which is reached after the present epoch, the nonadiabatic modes are constant and decaying.

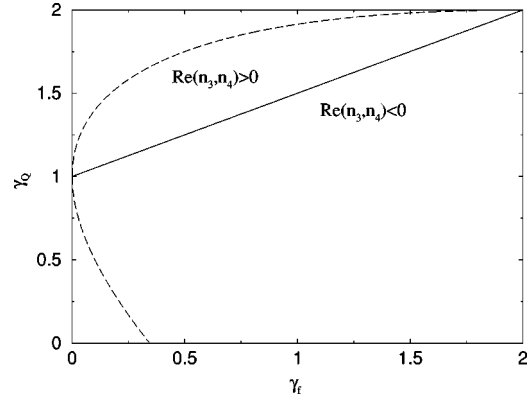


FIG. 2. Properties of the eigenvalues n_3 and n_4 in the case of a usual tracker regime, as a function of γ_f and γ_Q . Their real part is negative below the solid line and they have an imaginary part on the right-hand side of the dashed line.

IV. CONCLUSIONS

We have derived four first-order equations to describe large-scale perturbations in quintessence scenarios. During each of the five main regimes of quintessence behavior, these equations have constant coefficients, enabling analytic solution of the perturbations by eigenvector decomposition.

We have seen that during the kinetic regime there is a growing isocurvature mode which then remains constant until tracking begins. However, if the quintessence field undergoes a long period of tracking, there remain only adiabatic perturbations which, in the case of a subdominant tracker, are independent of its initial conditions. A low initial quintessence energy density [8], or a long kinetic period, may prevent the nonadiabatic modes from disappearing completely.

It is possible in principle to carry out the same analysis without using the large-scale approximation, although the equations may then be too complicated to be useful.

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